Journal of Engineering Mathematics **28**: 173–190, 1994.

On small-time expansion of nonlinear free surface problems

T. VINJE
Norwegian Contractors a.s, N-1320 Stabekk, Norway

Received 17 September 1990; accepted in revised form 12 August 1993

Abstract. The small-time expansion of the 2-D problem of a heaving semi-submerged circular cylinder, starting from rest, is analyzed. The first three terms of the small-time expansion of the outer solution (away from the cylinder) are developed. A logarithmic singularity is found for the second derivative in time of the surface elevation at the intersection point. This singularity is of second order in the heaving velocity of the cylinder, and the inner expansion developed for the *linear* wavemaker problem therefore does not cover this case. The problem of the horizontal motion of the semi-submerged cylinder is analyzed as well. For the case with the initial condition for the velocity potential chosen as zero on the free surface, the logarithmic singularity found for the wavemaker is recovered. Selecting the initial condition as a vanishing normal velocity on the free surface leads to a solution where the singularity appears in the fifth derivative in time of the free-surface elevation. The solution of this problem shows the characteristic behavior of formation of asymmetry, which finally would lead to formation of lee waves. The inner solution of the problem of the heaving cylinder is discussed, showing that the oscillatory behavior reported for the wavemaker problem appears in the present solution as well.

Introduction

During the last ten to fifteen years numerical simulation of nonlinear free surface problems, formulated in terms of potential theory, and solved according to the semi-Lagrangian scheme introduced by Longuet-Higgins and Cokelet [1], has been thoroughly investigated, and has been applied to a number of problems. The subject of the present paper is not this simulation procedure as such, but rather an approach to some of the problems related to the choice of initial values when starting the simulation process. Furthermore, the discussion will be limited to the occurrence of possible singularities at the intersection point between the free surface and a surface-piercing body. These singularities appear when formulating the solution in terms of a Taylor series in time, with space-dependent coefficients, which actually corresponds to keeping the spacial coordinates fixed, while letting the time approach zero. This, in turn, can be regarded as the outer solution of the problem, valid far away from the intersection point. It is established in the literature (Peregrine [2], Chwang [3], Lin [4]) that the singular behavior for the 2-D wavemaker, starting impulsively from rest, is given in terms of a $\log(z)$ singularity in the complex velocity $(u - iv)$ at the intersection point. The fact that this singularity grows increasingly worse for the higher derivatives in time confirms that this solution forms an outer expansion. Chwang [3] claims, on the basis of his Eulerian description, that the singularity will not be present in the fluid domain after a finite time has elapsed. This statement can be rejected on the basis that Newman's (see Lin [4]) analysis, which is based on a nonlinear Lagrangian scheme, gives the same singular behavior for the *particle* at the intersection point.

The singularity appears when the time is allowed to approach zero, while the spacial coordinates are of finite order, and *then* letting the spacial point approach the intersection point. In other terms: it appears as the inner expansion of the outer solution. Roberts [5], and later Joo et al. [6], have solved the *linear* initial value problem for the 2-D wavemaker, valid in the entire domain, and have shown that the inner solution is found in terms of a smooth function with rapidly oscillating wiggles in time (and/or space) on top of this curve, and that the outer expansion reflects the logarithmic singularity. Joo et al. [6] found that the wiggles of the inner solution were suppressed when surface tension was introduced. Computing the inner solution is not the main subject of the present paper, but a discussion of the behavior of this solution, in view of the present findings, will be given at the end of the paper.

There are problems which show a considerably worse singular behavior for small times than the $\log(z)$ singularity at the intersection point. Heaving circular cylinders, initially less than half submerged, have a strong singularity at the intersection point (see for instance Milne-Thomson [7]) §6.51). The singularity becomes a "square-root singularity" (corresponding to the solution for the flat plate) as the cylinder initially is completely out of the water. Miloh [8] has shown that the corresponding "square-root singularity" appears in the three-dimensional problem of a heaving sphere. Greenhow [9] has approached the 2-D nonlinear entry and re-entry problem of the less-than-half submerged circular cylinder numerically, reproducing the experimentally observed jet ejected from the area close to the intersection point. The inner problem for these cases has not been approached yet, but it is unlikely that the singularity would be completely removed, as was the case for the 2-D wavemaker.

There are other configurations which, at first sight, seem well behaved from a singularity point of view. One is the second-order diffraction problem of a vertical circular cylinder (with radius a), for which a controversy about a possible singularity at the intersection curve between the free surface and the cylinder was settled by Miloh [10]. The general problem of the possible singularity has been treated by Sclavounos [11].

A second configuration is the initially semi-submerged circular cylinder in heaving motion starting from rest, which will be discussed in the present paper. In this case it seems that no particular problems appear; the boundary conditions at the two intersecting boundaries are in correspondence, and thus no singularity should be present, even to second order for the stationary problem (Sclavounos [11]). This configuration has been used as a test case for nonlinear simulation procedures (Faltinsen [12], Vinje and Brevig [13]) and the solution for small times is thus of some interest, and will be investigated in the following.

A third configuration, which also will be looked into, is the one with a semi-submerged circular cylinder starting a surging motion from rest. This problem corresponds roughly to that introduced by Grosenbaugh and Yeung [14] in their numerical nonlinear simulation of the bow wave problem.

The inner problem for the heaving cylinder will be briefly discussed, and at the end of the paper some concluding remarks about the impact of the present findings for numerical simulations will be given.

Heaving motion of the half submerged cylinder

The problem is formulated in terms of 2-D potential flow and as an initial value problem, with the motion starting impulsively from rest. The initial conditions are given as follows (see Fig. 1):

$$\Phi = 0 \quad \text{for} \quad y = 0 \quad \text{and} \quad r > a \tag{1}$$

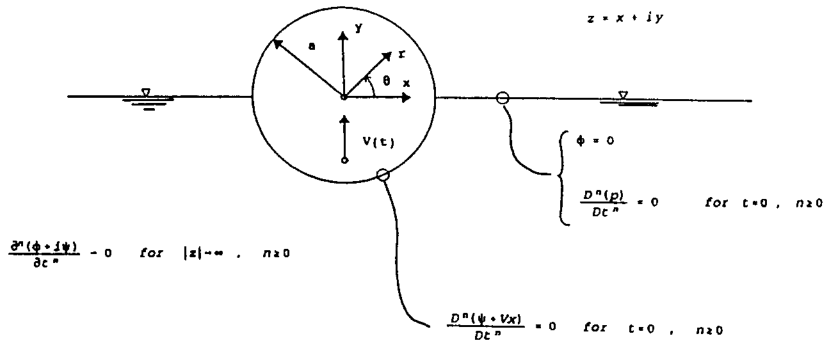

Fig. 1.

$$\Psi = -Vx \quad \text{for } r = a \text{ and } y < 0\,, \tag{2}$$

where V is the impulsive vertical velocity of the cylinder. The initial complex potential, $w_0(z)|_{t=0}$, is well known, obtained from the vertical dipole:

$$w_0(z) = \Phi_0 + i\,\Psi_0 = -i\,Va^2/z\,, \tag{3}$$

where $z = x + iy$.

Assuming that $w(z, t)$ is given by a Taylor series in time:

$$w(z, t) = w(z, 0) + \frac{\partial w(z, 0)}{\partial t}\,t + \frac{1}{2}\frac{\partial^2 w(z, 0)}{\partial t^2}\,t^2 + \cdots, \tag{4}$$

the corresponding boundary-value problems are found for the time derivatives at time $t = 0$ of the potential and of the stream function. The boundary conditions on the free surface are given by the dynamic condition as:

$$\frac{D^n(p)}{Dt^n} = 0\,, \quad n \geqslant 0\,, \quad \text{for} \quad y = 0 \quad \text{and} \quad r > a \quad \text{and} \quad t = 0\,, \tag{5}$$

which states that the pressure is zero (for small times) on this material surface. From the kinematic boundary condition on the cylinder we get the following:

$$\frac{D^n(\Psi + V(t)x)}{Dt^n} = 0 \quad \text{for} \quad r = a \quad \text{and} \quad y < 0\,. \tag{6}$$

The last equation is based on the assumption that once on the cylinder the fluid particles will remain on the cylinder. This is not necessarily the case, but is in correspondence with the assumption that the free surface forms a material surface. As an alternative, the Eulerian formulation, stating that $(\Psi + V(t)\cdot x)$ would not change when moving with the cylinder, could have replaced Eq. (6).

The condition that all the derivatives in time of $w(z, t)$ have to be analytic functions of z *inside* the fluid domain (but not necessarily on the boundary) is introduced as well. Furthermore, their derivatives with respect to z have to vanish faster than $1/z$ at infinity.

In this formation the kinematic boundary condition seems to be lacking. Actually this condition was introduced through the assumption that the free surface is a material surface. In its mathematical form it is only needed when analyzing the free-surface elevation. For the first derivative in time the following boundary conditions are found:

$$\frac{\partial \Phi}{\partial t} = -\tfrac{1}{2} v^2 = -V^2 \frac{a^4}{2x^4} \quad \text{for} \quad y = 0 \quad \text{and} \quad r > a \tag{7}$$

and

$$\frac{\partial \Psi}{\partial t} = -\frac{\mathrm{d}V}{\mathrm{d}t} a \cos(\theta) - V^2 \sin(2\theta) \quad \text{for } r = a \quad \text{and} \quad y < 0 \,. \tag{8}$$

The angle, θ, is defined in Fig. 1.

Before proceeding to the solution for the actual problem, the condition at the intersection point is investigated somewhat closer. The dynamic boundary condition on the free surface does actually correspond to zero horizontal acceleration along the surface, $y = 0$: $Du/Dt = 0$. By regarding the corresponding condition related to the boundary of the cylinder we have:

$$\left.\frac{Du}{Dt}\right|_{y=0,r=a} = \frac{\partial u}{\partial t} + \frac{1}{2} \left.\frac{\partial (u^2 + v^2)}{\partial x}\right|_{y=0,r=a} . \tag{9}$$

Taking into account that the velocity square is given from Eq. (3) as:

$$u^2 + v^2 = \frac{V^2 a^4}{r^4} , \tag{10}$$

we get:

$$\frac{1}{2} \left.\frac{\partial (u^2 + v^2)}{\partial x}\right|_{y=0,r=a} = -2 \frac{V^2}{a} . \tag{11}$$

Furthermore, we can express $\partial u / \partial t$ in terms of the time derivative of the stream function along the cylindrical boundary as:

$$\left.\frac{\partial u}{\partial t}\right|_{r=a} = \frac{1}{a} \left.\frac{\partial^2 \Psi}{\partial \theta \, \partial t}\right|_{r=a,\theta=0} = -2 \frac{V^2}{a} , \tag{12}$$

leading to:

$$\frac{Du}{Dt} = -4 \frac{V^2}{a} , \tag{13}$$

when computed on the basis of the boundary conditions on the cylinder. Notice that in this case the same inconsistency appears for the acceleration as the one found for the velocity when regarding the 2-D wavemaker problem and for the second-order diffraction problem of the vertical cylinder. The $z \cdot \log(z)$ singularity is thus expected to appear for the time derivative of the complex potential for the present problem.

The actual solution of the boundary-value problem is developed along the following lines:

Assume that the time derivative of the potential is given on the free surface in the following form:

$$\frac{\partial \Phi}{\partial t} = F(x) \,. \tag{14}$$

This boundary condition is then automatically satisfied if the time derivative of the complex potential is written in the form:

$$\frac{\partial w(z,0)}{\partial t} = F(z) + iG(z) , \tag{15}$$

where $F(x)$ and $G(x)$ are real.

The boundary condition for $G(z)$ on the cylinder is now determined from the original condition and from $F(z)$, in terms of sine and cosine expressions in $m\theta$. Recalling that

$$\mathrm{Im}\left[\log\frac{z-a}{z+a}\right]_{|z|=a, y=0} = -\pi/2 , \tag{16}$$

a solution satisfying the boundary conditions on the cylinder and the homogeneous condition on the free surface can be developed. This solution is denoted by $iH(z)$ in the following.

The asymptotic expansion of $iH(z)$, as $|z|\rightarrow\infty$, is written as $ih(z/a)$. Subtracting the function

$$iK(z) = i\left(h\left(\frac{z}{a}\right) - h\left(\frac{a}{z}\right)\right) \tag{17}$$

(which satisfies the homogeneous boundary conditions on the two boundaries) from $iH(z)$, the required function, $iG(z)$ is found.

The solution for the time derivative of the complex potential is developed as follows:

$$\begin{aligned}\frac{\partial w(z,0)}{\partial t} = &-i\frac{\mathrm{d}V}{\mathrm{d}t}\cdot\frac{a^2}{z} - \frac{1}{2}\frac{V^2a^4}{z^4} - i\frac{V^2}{\pi}\left\{\frac{7}{3}\left[\frac{z}{a}-\frac{a}{z}\right] + \left[\left(\frac{z}{a}\right)^3 - \left(\frac{a}{z}\right)^3\right]\right\} \\ &- i\frac{V^2}{\pi}\left\{\left[\left(\frac{z}{a}\right)^2 - \left(\frac{a}{z}\right)^2\right] + \frac{1}{2}\left[\left(\frac{z}{a}\right)^4 - \left(\frac{a}{z}\right)^4\right]\right\}\log\frac{z-a}{z+a} .\end{aligned} \tag{18}$$

Notice that the $(z-a)\cdot\log(z-a)$ singularity appears in the expression, as indicated earlier. A check on this formula is done simply by convincing oneself that the boundary conditions Eqs. (7) and (8) are satisfied, together with the condition at infinity. The fact that $\partial w/\partial t$ is an analytic function ensures uniqueness. A corresponding check can be made on most of the formulae developed in the present paper.

For the second derivative in time of the complex potential the following boundary conditions are found:

$$\frac{\partial^2\Phi}{\partial t^2} = -2v\frac{\partial v}{\partial t} - gv \quad \text{for} \quad y=0 \quad \text{and} \quad r>a \tag{19}$$

and

$$\begin{aligned}\frac{\partial^2\Psi}{\partial t^2} = &-\frac{\mathrm{d}^2V}{\mathrm{d}t^2}x - 2u\frac{\mathrm{d}V}{\mathrm{d}t} - V\frac{\partial(\frac{1}{2}(u^2+v^2))}{\partial x} \\ &-(V+v)\frac{\partial u}{\partial t} + u\frac{\partial v}{\partial t} \quad \text{for} \quad r=a \quad \text{and} \quad y=0 .\end{aligned} \tag{20}$$

Notice that gravity now appears in the problem. The solution to this problem is given in

Appendix A. The most important aspect of this solution is that the singularity at the intersection point is of the $\log(z)$ type, together with a weaker one, of the $z \cdot (\log(z))^2$ type. This corresponds to the results found for the 2-D wavemaker by Lin [4], with the only difference that it now appears for the second derivative in time instead of for the first derivative. The basis for the results for the small-time expansion for the force, given by Vinje and Brevig [15], was a Laurent series expansion in z of the time derivatives of the complex potential. This solution hid the logarithmic singularity completely (the series expansion can be shown to converge to Eq. (18)), but due to the finite first derivative in time of the free-surface elevation, the time derivative of the force could be computed. Notice that the second derivative in time of the free surface elevation at the intersection point has to be taken into account to compute the second derivative of the force. This has a logarithmic singularity at that point, which, at best, requires that special care should be taken when evaluating the integral. Another question that might be raised is if the force can be computed from the *outer* solution of the problem. It is not obvious that this could be done, but if the inner solution is valid in a thin layer close to the cylinder (which is the basis for calling it "the inner solution") and basically close to the intersection point, application of ideas similar to those leading to Prandtl's boundary layer theory (about the pressure given from the outer solution) seems to make sense. The force acting on the cylinder is then computed as:

$$F_y(0) = 2\rho a \int_{-\pi}^{0} \left[\frac{\partial \Phi}{\partial t} + \tfrac{1}{2}(u^2 + v^2) + gy \right]_{r=a} \sin(\theta) \cdot \mathrm{d}\theta \tag{21}$$

and after introduction of Eq. (18):

$$F_y(0) = \tfrac{1}{2}\pi\rho g a^2 - \tfrac{1}{2}\pi\rho a^2 \frac{\mathrm{d}V}{\mathrm{d}t} + \frac{8}{15}\rho a V^2 , \tag{22}$$

which corresponds to the result of Vinje and Brevig [15]. The first term corresponds to the familiar buoyancy force, the second to the common inertia term, and the third is due to the nonlinear effects. Notice that the third team is positive, independent of the sign of the velocity.

Appendix A makes it clear why the computation of the time derivative of the force is omitted in the present investigation. The result from Vinje and Brevig [15] indicates that it is positive. Vinje and Brevig [13] were concerned about the difference between the results found by Faltinsen [12] and those presented by Vinje and Brevig ([13], [15]). The comparison is shown in Fig. 2. The difference between the two solutions seems to appear in the second derivative of the force at time equal to zero. This component would be computed on the basis of the third derivative in time of Φ. This has a $1/z$ singularity at the intersection point, at the same time as an integration has to take place up to a level which is computed on the basis of the second derivative in time of the elevation at that point. This integral most probably does not exist. In that case, the different ways Faltinsen and Vinje and Brevig treated the intersection point problem might very well explain the differences between the results. In that case the most probable answer to the question raised by Vinje and Brevig [13] is that neither Faltinsen nor Vinje and Brevig have computed the "correct" force. A solution of the "inner", nonlinear problem might resolve this question, but so far this solution has not been calculated; what has been published so far is based on linear theory.

The time derivative of the free surface elevation is computed as follows:

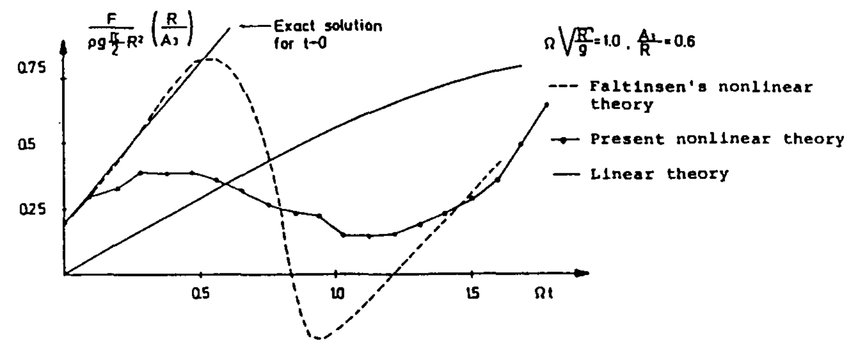

Fig. 2.

$$\frac{D\eta}{Dt}=\frac{\partial\eta}{\partial t}+u\frac{\partial\eta}{\partial x}=v \quad \text{for} \quad y=0\,, \quad x>a\,. \tag{23}$$

Taking into account the initial condition on the free surface we get:

$$\frac{\partial\eta}{\partial t}=v \quad \text{for} \quad y=0 \quad \text{and} \quad x>a\,. \tag{24}$$

For the second derivative we have:

$$\frac{D^2\eta}{Dt^2}=\frac{\partial^2\eta}{\partial t^2}=\frac{Dv}{Dt}=\frac{\partial v}{\partial t} \quad \text{for } y=0,\, x>a\,. \tag{25}$$

To arrive at the last expression, the initial conditions for Φ and for η have been introduced, together with the equation of continuity. The expressions in terms of x for the two time derivatives become:

$$\frac{\partial\eta}{\partial t}=-V\frac{a^2}{x^2} \quad \text{for} \quad y=0 \quad \text{and} \quad x>a \tag{26}$$

and

$$\begin{aligned}\frac{\partial^2\eta}{\partial t^2}=-\frac{\mathrm{d}V}{\mathrm{d}t}\frac{a^2}{x^2}+\frac{2V^2}{\pi x}\log\left(\frac{x-a}{x+a}\right)\cdot\left[\left(\frac{x}{a}\right)^4+\left(\frac{x}{a}\right)^2+\left(\frac{a}{x}\right)^2+\left(\frac{a}{x}\right)^4\right]\\ +\frac{4V^2}{3\pi x}\left(\frac{x}{a}+\frac{a}{x}\right)\left[3\left(\frac{x}{a}\right)^2+1+3\left(\frac{a}{x}\right)^2\right].\end{aligned} \tag{27}$$

Here we observe the expected logarithmic singularity at the intersection point.

Horizontal translation of a semi-submerged cylinder

Assuming that the semi-submerged circular cylinder starts impulsively from rest with a horizontal velocity U, the complex potential at $t=0$ is written:

$$w(z)=(\Phi+i\Psi)_{t=0}=\frac{Ui}{\pi}\left(z-\frac{a^2}{z}\right)\log\left(\frac{z-a}{z+a}\right), \tag{28}$$

which satisfies the condition $\Phi = 0$ on the free surface ($y = 0$) and $\Psi = Ua \cdot \sin(\theta)$ on the cylinder. This potential clearly has a $(z-a)\log(z-a)$ singularity at the intersection point between the free surface and the cylinder. Introducing the variable $\zeta = z - a$, and letting $a \to \infty$ while ζ is of order 1, the complex potential takes the form:

$$w(\zeta) \to \frac{2iU\zeta}{\pi} \log\left(\frac{\zeta}{2a}\right)\left[1 + O\left(\frac{\zeta}{2a}\right)\right], \tag{29}$$

which satisfies the condition $\Phi = 0$ for $y = 0$ and $\Psi = Uy$ for $(x = a, y < 0)$. These are the boundary conditions for the infinite-depth wavemaker, and the correspondence between the present solution and that of the 2-D wavemaker is established.

Grosenbaugh and Yeung [14] introduced a different initial condition for their numerical investigation of the bow-wave problem. They assumed that their 2-D body was at rest, being exposed to a current which was uniform at infinity. The free surface was assumed to be flat for $t = 0$. This corresponds to the condition $\Psi = 0$ for $y = 0$. In this case the intersection point is a stagnation point in the fluid, and no problems regarding singularities are expected. In the following this problem will be investigated somewhat further. For this purpose the geometry has been simplified to that of a semi-submerged circular cylinder. In contrast to the work of Grosenbaugh and Yeung, the free surface on both sides of the structure have been regarded. The cylinder has been assumed to be stationary, and being subject to a uniform, parallel flow at infinity. The geometry is given in Fig. 3, where the boundary conditions are given as well.

For this particular problem the value for $[\partial^n(\Phi + i\Psi)/\partial t^n]_{t=0}$ is given once the value of $\partial^n\Phi/\partial t^n|_{t=0} = F_n(x/a)$ is given on the free surface as:

$$\begin{aligned}\frac{\partial^n(\Phi + i\Psi)}{\partial t^n}_{t=0} &= F_n\left(\frac{z}{a}\right) - \frac{i}{\pi}\left[F_n\left(\frac{z}{a}\right) - F_n\left(\frac{a}{z}\right)\right]\log\left(\frac{z-a}{z+a}\right) \\ &\quad + \frac{i}{\pi}\left[K_n\left(\frac{z}{a}\right) - K_n\left(\frac{a}{z}\right)\right],\end{aligned} \tag{30}$$

where $K_n(z/a)$ is determined from the condition:

$$\lim_{|z| \to \infty}\left[F_n\left(\frac{a}{z}\right)\log\left(\frac{z-a}{z+a}\right) - K_n\left(\frac{a}{z}\right)\right] = 0 \tag{31}$$

and

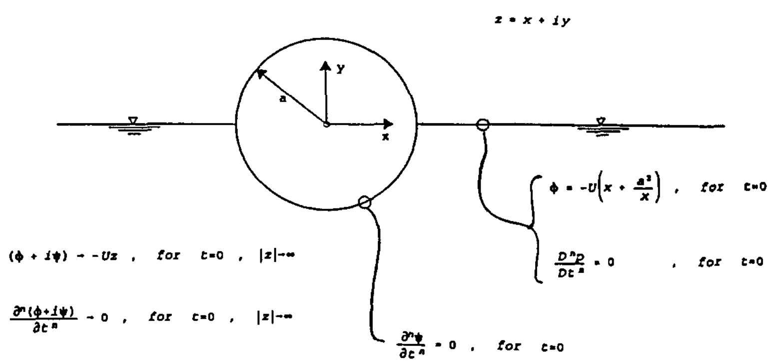

Fig. 3.

$$\lim_{|z|\to\infty} \left[K_n\left(\frac{a}{z}\right)\right] = 0 . \tag{32}$$

This solution requires that $\lim_{x\to\infty} F_n(x/a) = \text{constant}$ and that $F_n(z/a)$ does not have branchpoints at $z = \pm a$, which in turn means that the n-th derivative in time of the complex potential, computed according to the formulae, will not get a $\log(z-a)$ singularity, as long as $F_n(z/a)$ is analytic for $z = a$.

The complex potential for $t = 0$ is simply given as:

$$\Phi + i\Psi|_{t=0} = -Ua\left(\frac{z}{a} + \frac{a}{z}\right) . \tag{33}$$

The condition that $p = 0$ on the free surface then yields:

$$\frac{\partial \Phi}{\partial t}_{t=0,y=0} = -\tfrac{1}{2}u^2|_{t=0,y=0} = -\frac{U^2}{2}\left(1 - \left(\frac{a}{x}\right)^2\right)^2 \tag{34}$$

and accordingly:

$$K_1\left(\frac{z}{a}\right) = U^2\left[\left(\frac{z}{a}\right)^3 - \frac{5}{3}\frac{z}{a}\right], \tag{35}$$

which finally leads to:

$$\begin{aligned} \frac{\partial(\Phi + i\Psi)}{\partial t}_{t=0} &= -\tfrac{1}{2}U^2\left[1 - \left(\frac{a}{z}\right)^2\right]^2 \\ &\quad + i\frac{U^2}{2\pi}\left[\left(1 - \left(\frac{a}{z}\right)^2\right)^2 - \left(1 - \left(\frac{z}{a}\right)^2\right)^2\right]\log\left(\frac{z-a}{z+a}\right) \\ &\quad - i\frac{U^2}{\pi}\left[\left(\frac{z}{a}\right)^3 - \left(\frac{a}{z}\right)^3 - \frac{5}{3}\left(\frac{z}{a} - \frac{a}{z}\right)\right]. \end{aligned} \tag{36}$$

The complex potential does, as is easily shown, behave as follows:

$$\partial(\Phi + i\Psi)/\partial t|_{t=0} \underset{z\to a}{\to} -[8iU^2/\pi a^3]\,(z-a)^3 \log(z-a) . \tag{37}$$

For the next derivative, $\partial^2\Phi/\partial t^2|_{t=0,y=0}$, we get:

$$F_2(x/a) = \frac{\partial^2 \Phi}{\partial t^2}_{t=0,y=0} = -2u\frac{\partial u}{\partial t} - u^2\frac{\partial u}{\partial x} = u^2\frac{\partial u}{\partial x}, \tag{38}$$

where Eq. (34) has been utilized to arrive at the final result. Introducing the expression for the horizontal velocity then yields:

$$F_2\left(\frac{x}{a}\right) = -\frac{2U^3}{a}\left(\frac{a}{x}\right)^3\left(1 - \left(\frac{a}{x}\right)^2\right)^2 . \tag{39}$$

From this expression it is already seen that the second derivative of the complex potential with respect to time will have the same singular behaviour at $z = a$ as the first derivative, namely a $(z-a)^3\log(z-a)$ singularity. From Eq. (39) we develop:

$$K_2\left(\frac{z}{a}\right) = \frac{4U^3}{a}\left[\left(\frac{z}{a}\right)^6 - \frac{5}{3}\left(\frac{z}{a}\right)^4 - \frac{8}{15}\left(\frac{z}{a}\right) + \frac{8}{105}\right] \tag{40}$$

and accordingly we get for $\partial^2(\Phi + i\Psi)/\partial t^2|_{t=0}$:

$$\left.\frac{\partial^2(\Phi + i\Psi)}{\partial t^2}\right|_{t=0} = -\frac{2U^3}{a}\left(\frac{a}{z}\right)^3\left(1-\left(\frac{a}{z}\right)^2\right)^2 + \frac{2iU^3}{\pi a}\left[\left(\frac{a}{z}\right)^3\left(1-\left(\frac{a}{z}\right)^2\right)^2 - \left(\frac{z}{a}\right)^3\left(1-\left(\frac{z}{a}\right)^2\right)^2\right]\log\left(\frac{z-a}{z+a}\right) - \frac{4iU^3}{\pi a}\left[\left(\frac{z}{a}\right)^6 - \frac{5}{3}\left(\frac{z}{a}\right)^4 + \frac{8}{15}\left(\frac{z}{a}\right)^2 - \left(\frac{a}{z}\right)^6 + \frac{5}{3}\left(\frac{a}{z}\right)^4 - \frac{8}{15}\left(\frac{a}{z}\right)^2\right]. \tag{41}$$

So far the boundary condition on the free surface has been developed from the tangential velocity along this surface alone. For $F_3(x/a)$ this is not the case any longer. The boundary condition on the free surface for the third derivative in time is given as:

$$\left.\frac{\partial^3\Phi}{\partial t^3}\right|_{t=0} = -3u^2\left(\frac{\partial u}{\partial x}\right)^2 - u^3\left(\frac{\partial^2 u}{\partial x^2}\right) - 2\left(\frac{\partial v}{\partial t}\right)^2 - g\left(\frac{\partial v}{\partial t}\right). \tag{42}$$

Notice that for the first time gravity comes into effect, through the last term. To evaluate this expression, we have to determine $(\partial v/\partial t)$ on the free surface from Eq. (36). The result is given as:

$$\left.\frac{\partial v}{\partial t}\right|_{t=0,y=0} = -\frac{2U^2}{\pi a}\left\{\left[\left(\frac{x}{a}\right)^3 - \left(\frac{x}{a}\right) - \left(\frac{a}{x}\right)^3 + \left(\frac{a}{x}\right)^5\right]\log\left(\frac{x-a}{x+a}\right) + 2\left(\frac{x}{a}\right)^2 - \frac{4}{3} - \frac{4}{3}\left(\frac{a}{x}\right)^2 + 2\left(\frac{a}{x}\right)^4\right\}. \tag{43}$$

Due to the logarithmic term in Eq. (43), $F_3(z/a)$ does not satisfy the conditions required for application of Eq. (30). Special care has to be taken when $F_n(x/a)$ involves terms of the form:

$$\Gamma_n(x/a) = G_n(x/a)\log((x-a)/(x+a)), \tag{44}$$

where $G_n(z/a)$ is an analytic function, or of the form:

$$\Lambda_n(x/a) = R_n(x/a)\left[\log((x-a)/(x+a))\right]^2, \tag{45}$$

where $R_n(z/a)$ is an analytic function. The first expression will give a contribution to the n-th derivative of the complex potential in time of the following form:

$$\Gamma_n\left(\frac{z}{a}\right) - \frac{i}{2\pi}\cdot\left[G_n(z/a) - G_n\left(\frac{a}{z}\right)\right]\log\left(\frac{x-a}{x+a}\right) + \frac{i\pi}{4}\left[G_n\left(\frac{z}{a}\right) + G_n\left(\frac{a}{z}\right)\right] + i\left[H_n\left(\frac{z}{a}\right) - H_n\left(\frac{a}{z}\right)\right], \tag{46}$$

where $H_n(z/a)$ is introduced to satisfy the condition at infinity. The second expression leads to a contribution of the form:

$$\Lambda_n\left(\frac{z}{a}\right) + \frac{i\pi}{3}\left[2R_n\left(\frac{z}{a}\right) + R_n\left(\frac{a}{z}\right)\right]\log\left(\frac{z-a}{z+a}\right) + i\left[L_n\left(\frac{z}{a}\right) - L_n\left(\frac{a}{z}\right)\right] - \frac{i}{3\pi}\left[R_n\left(\frac{z}{a}\right) - R_n\left(\frac{a}{z}\right)\right]\left\{\log\left(\frac{z-a}{z+a}\right)\right\}^3, \tag{47}$$

where $L_n(z/a)$ is introduced to satisfy the condition at infinity.

By regarding Eqs. (43), (46) and (47), it becomes clear that the singular behavior of the third derivative of the complex potential with respect to time will be of the form:

$$(z^2 - a^2)^2 [\log((z-a)/(z+a))]^2 , \tag{48}$$

which, in turn, points to a $(z^2 - a^2)[\log((z-a)/(z+a))]^n$ singularity for the fourth derivative, where n takes the values 1, 2 or 3. This, in turn, means that the fourth derivative in time of the complex *velocity* will have a $\log((z-a)/(z+a))$ singularity. The singular behavior for this problem is thus much weaker than for the first problem discussed in the present paper.

The horizontal force acting on the cylinder can be expressed as follows:

$$F_x(t) = \rho a \int_{\theta_1(t)}^{\theta_2(t)} \cos(\theta) \left[\frac{\partial \Phi}{\partial t} + \tfrac{1}{2}(u^2 + v^2) + ga\sin(\theta) \right] \mathrm{d}\theta , \tag{49}$$

where $\theta_1(0) = -\pi$ and $\theta_2(0) = 0$. The integrand has to be computed at the cylinder surface.

We will now regard the force for $t = 0$. From the basic theory for potential flow around circular cylinders we can deduce that the contribution to the horizontal force from the velocity-square term is identically equal to zero. It is not surprising that the contribution from the static pressure is zero as well. From the fact that the "acting pressure" on the free surface (Eq. (34)) is symmetric about the y-axis we can conclude that $\partial\Phi/\partial t$ is symmetric about the same axis. (One can reach the same conclusion by examining Eq. (36).) This, in turn, means that the contribution to the horizontal force from $\partial\Phi/\partial t$ is zero. Finally, $F_x(0)$ is zero.

The vertical force is given as follows:

$$F_y(t) = \rho a \int_{\theta_1(t)}^{\theta_2(t)} \sin(\theta) \left[\frac{\partial \Phi}{\partial t} + \tfrac{1}{2}(u^2 + v^2) + ga\sin(\theta) \right] \mathrm{d}\theta . \tag{50}$$

The contribution from the static term is, as should be expected, equal to the buoyancy of the semi-submerged cylinder. The force from the velocity-square term is evaluated from the dipole potential and is formed to be $-8\rho aU^2/3$. The contribution from $(\partial\Phi/\partial t)$ becomes $8\rho aU^2/15$, which amounts to 1/5 of the contribution from the velocity-square term. The final result is:

$$F_y(t) = \frac{\pi}{2}\rho g a^2 - \frac{32}{15}\rho a U^2 . \tag{51}$$

The time derivative of the horizontal force is computed as follows:

$$\begin{aligned} \left.\frac{\mathrm{d}F_x(t)}{\mathrm{d}t}\right|_{t=0} &= -a\left[\frac{\mathrm{d}\theta_2}{\mathrm{d}t} p(\theta = \theta_2, r = a)\cos(\theta_2) - \frac{\mathrm{d}\theta_1}{\mathrm{d}t} p(\theta = \theta_1, r = a)\cos(\theta_1) \right]_{t=0} \\ &\quad - a\int_{-\pi}^{0} \cos(\theta) \left.\frac{\partial p}{\partial t}\right|_{t=0} \mathrm{d}\theta , \end{aligned} \tag{52}$$

where $(\mathrm{d}\theta_1/\mathrm{d}t)_{t=0}$ and $(\mathrm{d}\theta_2/\mathrm{d}t)_{t=0}$ are zero, due to the initial condition that the vertical velocity at the free surface is zero. Accordingly we have:

$$\left.\frac{\mathrm{d}F_x(t)}{\mathrm{d}t}\right|_{t=0} = \rho a \int_{-\pi}^{0} \left[\frac{\partial^2\Phi}{\partial t^2} + \frac{1}{2}\frac{\partial(u^2 + v^2)}{\partial t} \right]_{t=0, r=a} \cos(\theta)\,\mathrm{d}\theta . \tag{53}$$

The contribution to this integral from the time derivative of the velocity squared is zero. This can readily be seen from symmetry and asymmetry about the y-axis. The contribution to the integral from the log-related term of Eq. (40) is positive, but is more than balanced by the contribution from the term introduced to account for the conditions at infinity. The total integral becomes:

$$\left.\frac{\mathrm{d}F_x(t)}{\mathrm{d}t}\right|_{t=0} = -\frac{112}{315\pi}\rho U^3 = -0.113\,\rho U^3 . \tag{54}$$

The sign is negative, as should be expected, and *indicates* a resistance proportional to U^2 after a finite time has elapsed.

The free surface elevation is determined from the kinematic condition:

$$\left.\frac{D^n(\eta - y)}{Dt}\right|_{y=\eta} = 0 \tag{55}$$

and the initial condition:

$$\eta|_{t=0} = v|_{t=0,y=0} = 0 . \tag{56}$$

Accordingly we get for the derivatives of η in time:

$$\left.\frac{\partial\eta}{\partial t}\right|_{t=0} = 0 , \tag{57}$$

$$\left.\frac{\partial^2\eta}{\partial t^2}\right|_{t=0} = \left.\frac{\partial v}{\partial t}\right|_{y=0,t=0} , \tag{58}$$

and

$$\left.\frac{\partial^3\eta}{\partial t^3}\right|_{t=0} = \left[\frac{\partial^2 v}{\partial t^2} - \frac{\partial}{\partial x}\left(u\frac{\partial v}{\partial t}\right)\right]_{y=0,t=0} . \tag{59}$$

The term $(\partial^2\eta/\partial t^2)|_{t=0}$ is found directly from Eq. (4) as:

$$\left.\frac{\partial^2\eta}{\partial t^2}\right|_{t=0} = -\frac{2U^2}{\pi a}\left\{\left(\frac{a}{x}\right)\left[\left(\frac{x}{a}\right)^2 + 1 + \left(\frac{a}{x}\right)^2\right]\left(\frac{x}{a} - \frac{a}{x}\right)^2 \log\left[\frac{(x-a)}{(x+a)}\right]\right.$$
$$\left. + 2\left(\frac{x}{a}\right)^2 - \frac{4}{3} - \frac{4}{3}\left(\frac{a}{x}\right)^2 + 2\left(\frac{a}{x}\right)^4\right\} . \tag{60}$$

The second derivative of η in time is thus symmetric about the y-axis, which is consistent with $F_x|_{t=0}$ being zero. No tendency of the formation of lee waves is observed so far.

Let us now regard the expression for the third derivative of η in time (Eq. (59)). From the discussion about the symmetry relationship of $(\partial v/\mathrm{d}t)|_{y=0,t=0}$ given above, and from the fact that $u(x)$ is a symmetric function about the y-axis, the second term on the right-hand side of Eq. (59) is found to be asymmetric. From Eq. (40) it is clear that $(\partial^2 v/\partial t^2)_{y=0,t=0}$ will be asymmetric as well. Accordingly: $(\partial^3\eta/\partial t^3)|_{t=0}$ will be asymmetric, and creation of lee waves will now start to take place. This is consistent with $\mathrm{d}F_x(t)/\mathrm{d}t|_{t=0}$ being negative.

Comments on the inner solution of the problem of the heaving cylinder

As mentioned in the Introduction, the solution discussed above has to be regarded as the outer expansion (in space) when the time goes to zero. Consistent with the work by Roberts

[5] and by Joo et al. [6], the time development of the linearized problem for small time should be evaluated, and because the singularity is of second degree in the velocity, the time development of the *second order* problem should also be carried out. The first-order problem of the heaving cylinder has, in reality, been solved by Ursell [16], Maskell and Ursell [17] and Faltinsen [12]. This solution is based on the multipole expansion introduced by Ursell [18], and is developed for the forces acting on a heaving cylinder, and for the motion of a freely floating cylinder, starting impulsively from rest. The results are therefore not directly applicable to the present problem. In addition, the multipole expansion makes the solution less attractive for the asymptotic expansion for small times. To illustrate the effect, an "equivalent wavemaker" will be introduced (see for instance: Greenhow and Simon [19]). Since the solution of this "wavemaker" problem only *qualitatively* represents the heaving-cylinder problem, the solution will not be discussed in much detail, and only for the linear problem. Let us assume that a vertical boundary is situated at $x = a$ (see Fig. 4) and that the boundary condition at this surface is introduced by the horizontal velocity given for the heaving cylinder in Eq. (3):

$$u(y, t)_{x=a} = \frac{\partial \Psi}{\partial y} = \frac{2Va^3 y}{(a^2 + y^2)^2} . \tag{61}$$

The total flux through this surface is $(-Va)$, as should be expected, and the velocity normal to the vertical surface decreases from $u = 0$ at the free surface, reaches a minimum (of about $-0.65V$), and approaches zero for $y \rightarrow -\infty$. The small parameter in the present problem is the velocity V, which is assumed to be constant in time.

In contrast to what was done by Kennard [20], Roberts [5] and Joo et al. [6], the problem will be solved by means of a source distribution along the vertical boundary. The complex potential of a time-dependent source is given in Wehausen and Laitone [21], Eq. (13.54), which is written (after a misprint in sign has been corrected) as:

$$\begin{aligned} w(z, t; c) = {} & \frac{Q(t, c)}{2\pi} \log \left(\frac{z - c}{z + c^*} \right) \\ & - \frac{g}{\pi} \int_0^t Q(\tau, c)\, \mathrm{d}\tau \int_0^\infty (gk)^{-1/2} \sin[(gk)^{1/2}(t - \tau)] \exp(-ik(z - c^*))\, \mathrm{d}k\,, \end{aligned} \tag{62}$$

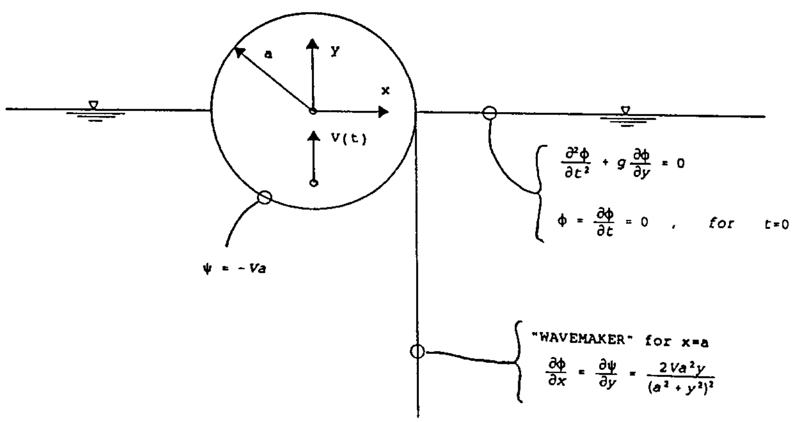

Fig. 4.

where $c = i\varsigma$ is the complex coordinate of the source point, c^* is the complex conjugate of c, and $Q(\tau, c)$ is the source intensity at $z = c$ at time τ. Once the stream function along the vertical boundary is known, the source intensity is found from continuity:

$$\frac{Q(t,c)}{2} = \frac{\partial \Psi(\varsigma, t)}{\partial y} = u(\varsigma, t)_{x=a} = u(\varsigma)\,, \tag{63}$$

where $u(\varsigma)$ is given in Eq. (61). The free-surface elevation is found as $\eta = -1/g \cdot (\partial\Phi/\partial t)_{y=0}$, which gives (corresponding to Kennard (1949)):

$$\eta = \frac{1}{\pi}\int_0^\infty \cos(kx) Q_L(ka)(gk)^{-1/2} \sin((gk)^{1/2} t)\,\mathrm{d}k\,, \tag{64}$$

where x is the local coordinate being zero at the line source. $Q_L(ka)$ is given as:

$$Q_L(ka) = \int_0^\infty Q(\zeta)\,\mathrm{e}^{-k\zeta}\,\mathrm{d}\zeta\,. \tag{65}$$

The outer expansion of η for small t is found by letting $t \to 0$ and keeping the leading term in the series expansion of the sine function. The coordinate x takes a finite value. The integral expression for $Q_L(ka)$ is introduced in Eq. (64), and the integration is first taken over k. After this integration has been performed, we are left with the expression:

$$\eta(x,t)|_{t\to 0+} = -\frac{4Va^3 t}{\pi}\int_0^\infty \frac{\zeta^2}{(a^2+\zeta^2)^2(x^2+\zeta^2)}\,\mathrm{d}\zeta\,. \tag{66}$$

This integral is evaluated by elementary methods, yielding:

$$\eta(x,t)|_{t\to 0} = -\frac{Va^2 t}{(a+x)^2}\,, \tag{67}$$

which corresponds to Eq. (26) (keeping in mind the difference in the definitions of x), and constitutes the first approximation to the outer expansion. Further approximations are found by introducing the remaining terms of the Taylor expansion of the sine function. The effect of gravity will be introduced in the next approximation.

The inner expansion is found by assuming that both x and t approach zero at the same time. To establish this solution, the integral will be studied in somewhat more detail. First of all we have:

$$Q_L(ka) = \int_0^\infty Q(\zeta)\,\mathrm{e}^{-k\zeta}\,\mathrm{d}\zeta = 2Va(ka\, f(ka) - 1)\,. \tag{68}$$

The function $f(z)$ is the auxiliary function of the cosine and sine integrals (Abramowitz and Stegun [22], Ch. 5):

$$f(z) = \int_0^\infty \exp(-zy)\frac{\mathrm{d}y_2}{1+y^2} = \mathrm{Ci}(z)\sin(z) - \mathrm{si}(z)\cos(z)\,, \tag{69}$$

giving $Q_L(ka)$ as a non-oscillatory function of ka. Introducing the variable $\alpha = (gk)^{1/2}t$ as a dummy variable in Eq. (64), one gets:

$$\eta(x, t) = \frac{2}{\pi g t} \int_0^\infty Q_L\left(\frac{\alpha^2 a}{g t^2}\right) \cos(\alpha^2 X) \sin(\alpha)\, d\alpha\ , \tag{70}$$

where $X = x/gt^2$ is a dimensionless variable which may, or may not, be of order one when t and/or x go to zero. Notice the correspondence between the expression for the free surface elevation in the present problem and the expression for the velocity potential for the Cauchy–Poisson problem as given by Lamb [23]. When $X \ll 1$ the integral of Eq. (70) can be recast in the form:

$$\eta(x, t) = \frac{2t}{\pi x} \int_0^\infty Q_L\left(\frac{\beta^2 a g t^2}{x^2}\right) \cos\left(\frac{\beta^2}{X}\right) \sin\left(\frac{\beta}{X}\right) d\beta \tag{71}$$

by introducing the variable $\beta = \alpha X$. The trigonometric functions combine into:

$$2\cos(\beta^2/X)\sin(\beta/X) = -\sin(R(\beta^2 - \beta)) + \sin(R(\beta^2 + \beta))\ , \tag{72}$$

where $R = 1/X \gg 1$, and the method of stationary phase (see: Erdelyi [24], Ch. 2.9) can be applied to the first term. The stationary point is $\beta = \frac{1}{2}$, and the leading-order contribution to the integral from this term is found to be:

$$\eta_1(x, t) = 4\left(\frac{V^2 a}{\pi g}\right)^{1/2} \left(\frac{x}{a}\right)^{3/2} \left(\frac{4x}{gt^2}\right)^2 \cos\left(\frac{gt^2}{4x} + \frac{\pi}{4}\right). \tag{73}$$

The function η_1 is oscillatory, with a strongly *decreasing* amplitude and "wavelength" as $x \to 0$. The oscillatory behavior is much like what was reported by Roberts [5] and Joo et al. [6]. The contribution from the second term of Eq. (72) is found in a different way. This contribution to the integral is expressed as follows:

$$\eta_2(x, t) = \frac{1}{2\pi (gx)^{1/2}} \int_0^\infty (R + \gamma)^{-1/2} \sin(\gamma) Q_L\left(\frac{a \cdot \gamma^2}{4xR}\right) d\gamma\ , \tag{74}$$

where the approximation:

$$((R + \gamma)^{1/2} - R^{1/2})^2 \approx \gamma^2/4R \tag{75}$$

is introduced, being valid for $\gamma/R \ll 1$. Assuming that (a/x) is of order $R^{3/2}$ the Q_L function will be small for finite values of γ. This, in turn, means that the integral can be approximated as follows:

$$\begin{aligned}\eta_2(x, t) &\sim \frac{1}{2\pi (gxR)^{1/2}} \int_0^\infty \gamma Q_L\left(\frac{a\gamma^2}{4Rx}\right) d\gamma \sim \frac{t}{\pi a} \int_0^\infty Q_L(z)\, dz \\ &= -\frac{2Vt}{\pi} \int_0^\infty (1 - zf(z))\, dz\ ,\end{aligned} \tag{76}$$

where $f(z)$ is given in Eq. (69). From the relations between the auxiliary functions of the cosine and sine integral (Abramowitz and Stegun [22], Ch. 5), the last integral can be shown to take the value $(\pi/2)$, and we finally get:

$$\eta_2(x, t) \sim -Vt ,$$

which is equal to the inner limit of the outer expansion for small t. This is not surprising in view of the absence of a singularity in the outer solution. Furthermore, the approximations made for the integral (Eq. (76)) are actually the same as assuming that both x and t are small in Eq. (64). This contribution, Eq. (77), is of order (1), and is valid for $(gt^2/4x) \gg 1$ and $x \ll a$. The oscillatory η_1 term is significantly smaller than η_2, which, in principle, corresponds to the solutions given by Roberts [5] and Joo et al. [6].

The interesting point regarding the outer solution for the heaving cylinder is that the $z \cdot \log(z)$ singularity appears for the derivative in time of the complex potential, and is quadratic in the "small" parameter, V. To investigate this solution further, the initial-value problem for the second-order expansion has to be found. In this case the wavemaker approximation *might* still make sense, but most probably the proper solution for the circular cylinder has to be investigated. The reason is that the inconsistency in the horizontal acceleration at the intersection point, which causes the singular behavior, is due to the curvature of the cylinder at that point. This information is, seemingly, absent in the wavemaker approximation.

Concluding remarks

The investigation of the initial-value problem of a semi-submerged circular cylinder, starting from rest, shows that the small-time outer expansion of the velocity potential has a $z \log(z)$ singularity at the intersection point between the free surface and body in one of the (Eulerian) time derivatives. For the heaving cylinder, starting impulsively from rest, it occurs in the first derivative. Starting the motion with an impulsive acceleration will only transfer the singularity to the second derivative. For a cylinder moving horizontally with an impulsive velocity, the same problem occurs as is the case for the 2-D wavemaker: the $z \log(z)$ singularity appears in the velocity potential. By introducing a uniform flow at infinity, starting "impulsively", the singularity occurs in a much higher derivative.

All investigations so far (Chwang [3], Lin [4], Wang and Chwang [25] and the present) seem to indicate that the outer expansion of the problem, formulated in terms of a Taylor series in time, with space-dependent coefficients, does not converge (at least not uniformly) at the surface of the body. The inner expansion (of the linear problem) is finite in the actual domain close to the body and shows an oscillatory behavior, with rapidly decreasing wave lengths when approaching the surface of the body.

There is no doubt from present experience that the semi-Lagrangian (and other) numerical simulation procedure actually simulates the outer solution of the problem. To what extent it is possible to recapture the rapidly oscillating inner solution by use of the presently available nonlinear simulation tools is a moot question. If it makes any sense trying to do so is another question. The "wiggles" of the inner solution are not observed in model tests (see: Lin [4] and Joo et al. [6]) have shown that they are removed by introducing surface tension. They have further shown that introduction of surface tension does not alter the singular behavior of the outer expansion (see: Lin [4] and Joo et al. [6]), and surface tension is thus of little help in "improving" the numerical schemes. It would seem that singularities remain an integral part of the potential when trying to integrate the solution numerically.

From the discussion in the present paper the question might be raised if the singularity of the time derivative, for otherwise well-behaved problems, would be reflected in the numerical solution. No problems of this kind were reported by Faltinsen [12] or by Vinje and Brevig [13] when solving numerically the heaving-cylinder problem. This does not mean that the singularity in the time derivative will not appear. The results reported by Faltinsen and by Vinje and Brevig were found for rather large panel sizes on the free surface, which may have suppressed the development of the singularity. Furthermore, both procedures involved a certain spatial smoothing in the determination of the velocity at the free surface. This velocity was used when integrating the kinematic boundary condition to move the free-surface elevation forward in time. It would be of certain interest to have verified if, or if not, the singularity in the higher derivatives would be reflected in the numerical simulations. Furthermore, it would be of some interest to have the results of Fig. 2 recomputed, to see if the time history of the force depends on the treatment of the intersection point. In view of the development in computer technology since 1976–1980, a proper investigation of these problems would not be as difficult a task today as it was then.

Appendix A. The solution of the problem defined by Eqs. (19) and (20)

The solution consists of the three contributions, defined from:

(1) The contribution due to the inhomogeneous free-surface condition.
(2) The contribution due to the inhomogeneous body boundary condition.
(3) The contribution due to the conditions at infinity.

The three contributions are denoted w_{1tt}, w_{2tt} and w_{3tt} respectively. The three contributions are found to be:

$$w_{1tt} = -\frac{2V\dot{V}a^4}{z^4} + \frac{gVa^2}{z^2} + \frac{8V^3a^2}{3\pi z^3}\left(\frac{z}{a}+\frac{z}{z}\right)\left[3\left(\frac{z}{a}\right)^2 + 1 + 3\left(\frac{a}{z}\right)^2\right]$$
$$+\frac{4V^3a^2}{\pi z^3}\log\left(\frac{z-a}{z+a}\right)\left[\left(\frac{z}{a}\right)^4 + \left(\frac{z}{a}\right)^2 + \left(\frac{a}{z}\right)^2 + \left(\frac{a}{z}\right)^4\right], \tag{A.1}$$

$$w_{2tt} = -\left(\frac{\mathrm{d}^2V}{\mathrm{d}t^2}\right)\frac{ia^2}{z} - \frac{iV\dot{V}}{\pi}\log\left(\frac{z-a}{z+a}\right)\left[2\left(\frac{z}{a}\right)^4 - 2\left(\frac{a}{z}\right)^4 + 3\left(\frac{z}{a}\right)^3 - 3\left(\frac{a}{z}\right)^3\right]$$
$$+\frac{iV^3}{a}\left\{\left(\frac{z}{a}\right)^7 + \left(\frac{z}{a}\right)^5 + \left(\frac{z}{a}\right)^3 + 3\left(\frac{z}{a}\right) + 3\left(\frac{a}{z}\right) + \left(\frac{a}{z}\right)^3 + \left(\frac{a}{z}\right)^5 + \left(\frac{a}{z}\right)^7\right\}$$
$$+\frac{8iV^3}{3\pi^2a}\log\left(\frac{z-a}{z+a}\right)\left\{3\left(\frac{z}{a}\right)^6 + 4\left(\frac{z}{a}\right)^4 + 4\left(\frac{z}{a}\right)^2 - 4\left(\frac{a}{z}\right)^2 - 4\left(\frac{a}{z}\right)^4 - 3\left(\frac{a}{z}\right)^6\right\}$$
$$+\frac{2iV^3}{\pi^2a}\left[\log\left(\frac{z-a}{z+a}\right)\right]^2\left\{\left(\frac{z}{a}\right)^7 + \left(\frac{z}{a}\right)^5 - \left(\frac{a}{z}\right)^5 - \left(\frac{a}{z}\right)^7\right\}, \tag{A.2}$$

$$w_{3tt} = -\frac{2iV\dot{V}}{\pi}\left[2\left(\frac{z}{a}\right)^3 + \frac{11}{3}\left(\frac{z}{a}\right) - \frac{11}{3}\left(\frac{a}{z}\right) - 2\left(\frac{a}{z}\right)^3\right]$$
$$-\frac{iV^3}{a}\left[\left(\frac{z}{a}\right)^7 + \left(\frac{z}{a}\right)^5 + \left(\frac{z}{a}\right)^3 + 3\left(\frac{z}{a}\right) - 3\left(\frac{a}{z}\right) - \left(\frac{a}{z}\right)^3 - \left(\frac{a}{z}\right)^5 - \left(\frac{a}{z}\right)^7\right]$$
$$+\frac{8iV^3}{3\pi a}\left[3\left(\frac{z}{a}\right)^5 + 5\left(\frac{z}{a}\right)^3 + \frac{25}{3}\left(\frac{z}{a}\right) - \frac{25}{3}\left(\frac{a}{z}\right) - 5\left(\frac{a}{z}\right)^5 - 3\left(\frac{a}{z}\right)^7\right]. \tag{A.3}$$

References

1. M.S. Longuet-Higgins and E.D. Cokelet, The deformation of steep surface waves on water, I. A numerical method for computation. *Proc. Roy. Soc. London* A350 (1976) 1–28.
2. D.H. Peregrine, Flow due to vertical plate moving in a channel. Unpublished notes (1972).
3. A.T. Chwang, Nonlinear hydrodynamic pressure on accelerating plate. *Phys. Fluids* 26 (1983) 383–387.
4. W.-M. Lin, *Nonlinear Motion of the Free Surface near a Moving Body*. Ph.D. Thesis, Dept. Ocean Eng., Mass. Inst. Techn., Cambridge, Mass., U.S.A. (1984).
5. A.L. Roberts, Transient free-surface flows generated by a moving vertical plate. *Q. J. Mech. Appl. Math.* 40 (1987) 129–158.
6. S.W. Joo, W.W. Schultz and A.F. Messiter, Uniformly valid solutions to the initial-value wavemaker problem. Rept. No. 89-01, Univ. Mich. Prog. in Ship Hydrodynamics, Coll. Eng., Univ. Mich., Ann Arbor, Mich., U.S.A. (1989).
7. L.M. Milne-Thomson, *Theoretical Hydrodynamics*, MacMillan, London (1968).
8. T. Miloh, On the initial stage slamming of a rigid sphere in a vertical water entry. *Appl. Ocean Res.* 13 (1991) 43–48.
9. M.J. Greenhow, Water-entry and -exit of a horizontal cylinder. *Appl. Ocean Res.* 10 (1988) 191–198.
10. T. Miloh, Irregularities in solutions of nonlinear wave diffraction problem by vertical cylinder. *J. Waterway, Port, Coastal and Ocean Div.* 106 (1980) 279–284.
11. P.D. Sclavounos, Radiation and diffraction of second-order surface waves by floating bodies. *J. Fluid Mech.* 196 (1988) 65–91.
12. O.M. Faltinsen, Numerical solutions of transient nonlinear free-surface motion outside and inside moving bodies. In: 2nd Int. Conf. Num. Ship Hydrodyn., Berkeley, Cal., U.S.A. (1985).
13. T. Vinje and P. Brevig, Nonlinear ship motion. In: 3rd Int. Conf. Num. Ship Hydrodyn., Paris, France (1981).
14. M.A. Grosenbaugh and R.W. Yeung, Nonlinear bow flow – An experimental and theoretical investigation. In: 17th ONR Symp. Naval Hydrodyn., The Hague, The Netherlands (1988).
15. T. Vinje and P. Brevig, Nonlinear, two-dimensional ship motion. SIS Rep., Norwegian Hydrodyn. Lab., Trondheim, Norway (1980).
16. F. Ursell, The decay of the free motion of a floating body. *J. Fluid Mech.* 19 (1964) 305–319.
17. S.J. Maskell and F. Ursell, The transient motion of a floating body. *J. Fluid Mech.* 44 (1970) 303–313.
18. F. Ursell, On the heaving motion of a circular cylinder on the surface of a fluid. *Q. J. Mech. Appl. Math.* 2 (1949) 218–231.
19. M.J. Greenhow and M. Simon, A note on the equivalent wavemaker method. *Appl. Ocean Res.* 7 (1985) 107–112.
20. E.H. Kennard, Generation of surface waves by a moving partition. *Q. J. Appl. Math.* 7 (1949) 303–312.
21. J.V. Wehausen and E.V. Laitone, *Surface waves*. In: Handbuch der Physik, Vol. 9, Springer Verlag, Berlin, Germany (1960).
22. M. Abramowitz and I.E. Stegun, *Handbook of mathematical functions*, 5th Edition, Dover, New York, U.S.A. (1968).
23. H. Lamb, *Hydrodynamics*, Cambridge Univ. Press, Cambridge, U.K. (1932).
24. A. Erdelyi, *Asymptotic expansions*, Dover, New York, U.S.A. (1956).
25. K.-H. Wang and A.T. Chwang, Nonlinear free-surface flow around an impulsively moving cylinder. *J. Ship Res.* 33 (1989) 194–202.